

# Around Sperner's lemma

Oleg R. Musin\*

## Abstract

We consider a generalization of the classic Sperner lemma. This lemma states that every Sperner coloring of a triangulation of a simplex contains a fully colored simplex. We found a weaker assumption than Sperner's coloring. It is also shown that the main theorem implies Tucker's lemma and some other theorems.

**Keywords:** Sperner's lemma, Tucker's lemma, degree of mapping

## 1 Introduction

Throughout this paper the symbol  $\mathbb{R}^d$  denotes the Euclidean space of dimension  $d$ . We denote by  $\mathbb{B}^d$  the  $d$ -dimensional ball and by  $\mathbb{S}^d$  the  $d$ -dimensional sphere. If we consider  $\mathbb{S}^d$  as the set of unit vectors  $x$  in  $\mathbb{R}^{d+1}$ , then points  $x$  and  $-x$  are called *antipodal* and the symmetry given by the mapping  $x \rightarrow -x$  is called the *antipodality* on  $\mathbb{S}^d$ .

### 1.1 Sperner's lemma

*Sperner's lemma* is a statement about labellings (colorings) of triangulated simplices ( $d$ -balls). It is a discrete analog of the Brouwer fixed point theorem.

Let  $S$  be a  $d$ -dimensional simplex with vertices  $v_1, \dots, v_{d+1}$ . Let  $T$  be a triangulation of  $S$ . Suppose that each vertex of  $T$  is assigned a unique label from the set  $\{1, 2, \dots, d+1\}$ . A labelling  $L$  is called *Sperner's* if the vertices are labelled in such a way that a vertex of  $T$  belonging to the interior of a face  $F$  of  $S$  can only be labelled by  $k$  if  $v_k$  is on  $S$ .

**Theorem 1.1. (Sperner's lemma [13])** *Every Sperner labelling of a triangulation of a  $d$ -dimensional simplex contains a cell labelled with a complete set of labels:  $\{1, 2, \dots, d+1\}$ .*

---

\*This research is partially supported by NSF grant DMS - 1101688.

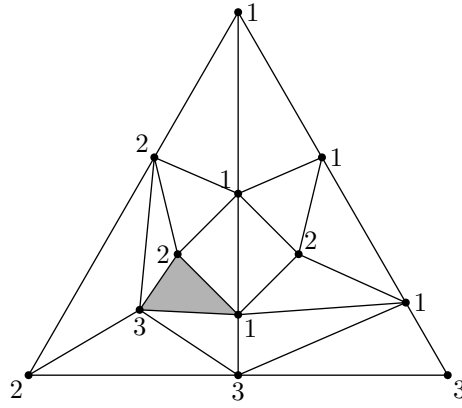


Figure 1: A 2-dimensional illustration of Sperner's lemma

The two-dimensional case is the one referred to most frequently. It is stated as follows:

*Given a triangle  $ABC$ , and a triangulation  $T$  of the triangle. The set  $V(T)$  of vertices of  $T$  is colored with three colors in such a way that*

- (i)  $A, B$  and  $C$  are colored 1, 2 and 3 respectively*
- (ii) Each vertex on an edge of  $ABC$  is to be colored only with one of the two colors of the ends of its edge. For example, each vertex on  $AC$  must have a color either 1 or 3.*

*Then there exists a triangle from  $T$ , whose vertices are colored with the three different colors.*

Consider a convex polytope  $P$  in  $\mathbb{R}^d$  defined by  $n$  vertices  $v_1, \dots, v_n$ . Let  $T$  be a triangulation of  $P$ , and suppose that the vertices of  $T$  have a labelling satisfying these conditions: each vertex of  $P$  is assigned a unique label from the set  $\{1, 2, \dots, n\}$  and each other vertex  $v$  of  $T$  is assigned a label of one of the vertices of  $P$  in the *carrier* of  $v$  that is the smallest face  $F$  of  $P$  that contains  $v$ . Such a labelling is called a *Sperner labelling* of  $T$ . We say that a  $d$ -simplex in the triangulation  $T$  is a *fully labelled (or colored) simplex* if all its labels are distinct.

There are several extensions of Sperner's lemma. One of the most interesting is the De Loera - Petersen - Su theorem. In [3] they proved the Atanassov conjecture [1].

**Theorem 1.2. (Polytopal Sperner's lemma [3])** *Let  $P$  be a convex polytope in  $\mathbb{R}^d$  with  $n$  vertices. Let  $T$  be a triangulation of  $P$ . Let  $L : V(T) \rightarrow \{1, 2, \dots, n\}$  be a Sperner labelling. (Here  $V(T)$  denote the set of vertices of  $T$ .) Then there are at least  $(n - d)$  fully-colored  $d$ -simplices of  $T$ .*

## 1.2 Tucker's lemma

Let  $T$  be a triangulation of the  $d$ -dimensional ball  $\mathbb{B}^d$ . We call  $T$  *antipodally symmetric on the boundary* if the set of simplices of  $T$  contained in the boundary of  $\mathbb{B}^d = \mathbb{S}^{d-1}$  is an

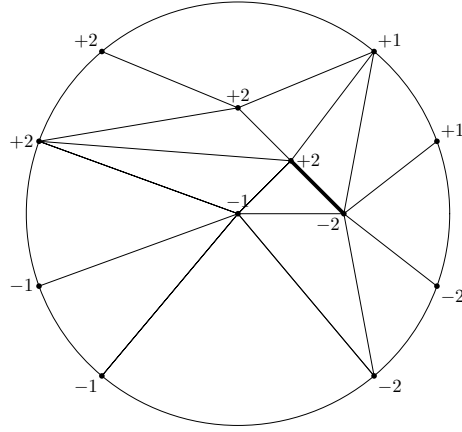


Figure 2: A 2-dimensional illustration of Tucker's lemma

antipodally symmetric triangulation of  $\mathbb{S}^{d-1}$ , that is if  $s \subset \mathbb{S}^{d-1}$  is a simplex of  $T$ , then  $-s$  is also a simplex of  $T$ .

**Theorem 1.3. (Tucker's lemma [14])** *Let  $T$  be a triangulation of  $\mathbb{B}^d$  that antipodally symmetric on the boundary. Let*

$$L : V(T) \rightarrow \{+1, -1, +2, -2, \dots, +d, -d\}$$

*be a labelling of the vertices of  $T$  that satisfies  $L(-v) = -L(v)$  for every vertex  $v$  on the boundary. Then there exists an edge in  $T$  that is **complementary**, i.e. its two vertices are labelled by opposite numbers.*

Tucker's lemma was extended by Ky Fan [5]:

**Theorem 1.4.** *Let  $T$  be a triangulation of  $\mathbb{B}^d$  that antipodally symmetric on the boundary. Let*

$$L : V(T) \rightarrow \{+1, -1, +2, -2, \dots, +n, -n\}$$

*be a labelling of the vertices of  $T$  that satisfies  $L(-v) = -L(v)$  for every vertex  $v$  on the boundary. Suppose this labelling does not have complementary edges. Then there are an odd number of  $d$ -simplices of  $T$  whose labels are of the form  $\{k_0, -k_1, k_2, \dots, (-1)^d k_d\}$ , where  $1 \leq |k_0| < |k_1| < \dots < |k_d| \leq n$  and all  $k_i$  have the same sign. In particular,  $n \geq d + 1$ .*

### 1.3 Main results and examples in dimension two

In this paper we consider an extension of Sperner's lemma, see Theorem 2.1 for two dimensions and Theorem 4.1 for the general case, that also yields extensions of a polytopal Sperner's lemma, Tucker's lemma and Ky Fan's lemma. Consider here only two-dimensional corollaries given in Section 3.

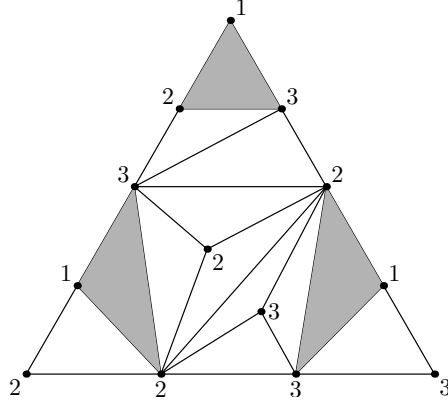


Figure 3:  $\deg(L, \partial T) = 3$ . There are three fully labelled triangles, see Corollary 3.1.

Let  $L : V \rightarrow \{1, 2, 3\}$  be a labelling of a set  $V := \{v_1, \dots, v_m\}$  in a circle. Let

$$\deg([1, 2], L) := p_* - n_*,$$

where  $p_*$  (respectively,  $n_*$ ) is the number of (ordering) pairs  $(v_k, v_{k+1})$  such that  $L(v_k) = 1$  and  $L(v_{k+1}) = 2$  (respectively,  $L(v_k) = 2$  and  $L(v_{k+1}) = 1$ ).

For instance, let  $L = (1221231232112231231)$ . Then  $p_* = 5$  and  $n_* = 2$ . Thus,  $\deg([1, 2], L) = 5 - 2 = 3$ .

Note that if instead of  $[1, 2]$  we take  $[2, 3]$  or  $[3, 1]$ , then we get that  $\deg([1, 2], L) = \deg([2, 3], L) = \deg([3, 1], L)$ , see Lemma 2.1.

Let  $T$  be a triangulation of a simple polygon  $M$ . Denote by  $\partial T$  the boundary of  $T$ . Then  $\partial T$  is a polygonal contour with vertices  $v_1, \dots, v_m$  that can be considered as points in a circle. We assume that these vertices are in counterclockwise order.

Let  $L : T \rightarrow \{1, 2, 3\}$  be a labelling. This labelling implies the labelling  $L_0 : \partial T \rightarrow \{1, 2, 3\}$ . Denote  $\deg(L, \partial T) := \deg([1, 2], L_0)$ .

**Corollary 3.1.** *Let  $T$  be a triangulation of a planar polygon  $M$ . Then for any labelling  $L : V(T) \rightarrow \{1, 2, 3\}$   $T$  must contain at least  $|\deg(L, \partial T)|$  fully colored triangles.*

This corollary is extended for all dimensions in Corollary 4.1 for oriented manifolds and in Corollary 4.5 for non-orientable manifolds.

Consider two examples. In Fig. 1 is given a Sperner's labelling. For this case  $\deg(L, \partial T) = 1$ . Therefore, Corollary 3.1 yields that there exist at least one triangle with labels 1, 2, 3.

In Fig. 3 is shown a labelling with  $\deg(L, \partial T) = 3$ . There we have exactly three fully labelled triangles.

Actually, we can extend all results for the case when  $M$  is a polygon with holes. In Definition 2.2 is considered this case.

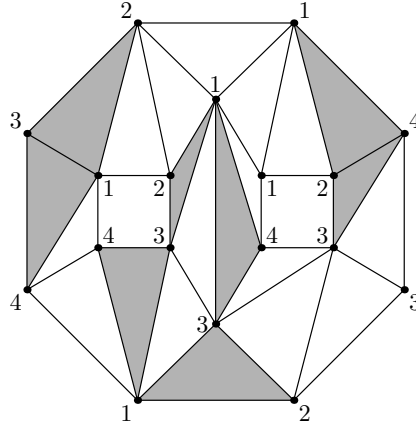


Figure 4: Octagon with two square holes. Here  $n = 4$ ,  $\deg(L, \partial T) = 4$  and there are eight fully labelled triangles, see Corollary 3.3.

In Section 2 also considered  $n$ -labellings  $L : V \rightarrow \{1, 2, \dots, n\}$ . For  $n > 3$  we consider only neighboring labellings NL, see Definition 2.1. Let  $T$  be a triangulation of  $M$ . We write that  $L \in \text{NL}(T, n)$ , where  $L : V(T) \rightarrow \{1, 2, \dots, n\}$ , if  $L$  is NL for any connected component of  $\partial T$ .

**Corollary 3.3.** *Let  $T$  be a triangulation of a planar polygon  $M$ . Let  $L$  be a labelling such that  $L \in \text{NL}(T, n)$ . Then the number of fully labelled triangles in  $T$  is at least  $(n-2)|\deg(L, \partial T)|$ .*

Corollary 4.2 extends this corollary for all dimensions.

In Fig. 4 we present a labelling with  $n = 4$  and  $\deg(L, \partial T) = 4$ . In this example there are exactly eight fully labelled triangles.

Tucker's and Ky Fan's lemmas also follow from Theorem 2.1, see Corollaries 3.4 and 3.5. Consider an extension of Tucker's lemma. If we have a labelling  $L : V(T) \rightarrow \{+1, -1, +2, -2\}$ , then it can be considered as a 4-labelling with the correspondence  $1 \rightarrow 1$ ,  $2 \rightarrow 2$ ,  $-1 \rightarrow 3$ , and  $-2 \rightarrow 4$ .

**Corollary 3.4.** *Let  $T$  be a triangulation of  $\mathbb{B}^2$  that antipodally symmetric on the boundary. Let  $L : V(T) \rightarrow \{+1, -1, +2, -2\}$  be a labelling that is antipodal on the boundary. Suppose there are no complementary edges on the boundary. Then there are at least  $|\deg(L, \partial T)|$  internal complementary edges. In particular, there is at least one.*

(Note that Lemma 3.1 states that  $\deg(L, \partial T)$  is odd.)

In Fig. 2 is given a labelling with  $\deg(L, \partial T) = 1$  and exactly one complementary edge. In Fig. 5 we have  $\deg(L, \partial T) = 3$  and three complementary edges.

Higher dimensional versions of Corollaries 3.4 and 3.5 are Corollaries 4.3 and 4.4.

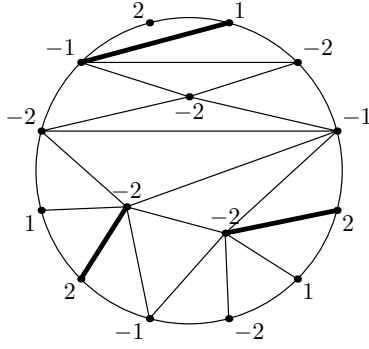


Figure 5: Since  $\deg(L, \partial T) = 3$ , there are three complementary edges (Corollary 3.4).

## 2 A generalization of the planar Sperner lemma

**Definition 2.1.** Let labels  $\{1, 2, \dots, n\}$  be in cyclic order i. e.  $i$  and  $i + 1$  as well as  $n$  and  $1$  are neighbors. Let  $V := \{v_1, \dots, v_m\}$  be points in a circle. We call  $L : V \rightarrow \{1, 2, \dots, n\}$  a *neighboring labelling* and write  $L \in \text{NL}(m, n)$  if for all vertices  $v_i$  and  $v_{i+1}$ , where  $v_{m+1} = v_1$ , either  $L(v_i) = L(v_{i+1})$  or  $L(v_i)$  and  $L(v_{i+1})$  are neighbors.

Let for  $L \in \text{NL}(m, n)$ ,

$$\deg([i, i + 1], L) := p(i) - n(i),$$

where  $p(i)$  (respectively,  $n(i)$ ) is the number of (ordering) pairs  $(v_k, v_{k+1})$  such that  $L(v_k) = i$  and  $L(v_{k+1}) = i + 1$  (respectively,  $L(v_k) = i + 1$  and  $L(v_{k+1}) = i$ ).

Similarly  $\deg([i + 1, i], L)$  can be defined. It is clear that

$$\deg([i + 1, i], L) = -\deg([i, i + 1], L).$$

Note that any 3-labelling  $L \in \text{NL}(m, 3)$ .

**Lemma 2.1.** Let  $L \in \text{NL}(m, n)$ . Then

$$\deg([1, 2], L) = \dots = \deg([n - 1, n], L) = \deg([n, 1], L).$$

*Proof.* Denote  $s_i := L(v_{i+1}) - L(v_i)$  for  $i = 1, \dots, m$ . Then  $s_i = -1, 0$  or  $1$ . Let  $S_k := s_1 + \dots + s_k$ . Now we prove that

$$S_k = L(v_{k+1}) - L(v_1) + q_k n,$$

where  $q_k$  is integer.

Indeed, without loss of generality we may assume that  $L(v_1) = 1$ . Let  $i$  is the next index in the sequence  $1, 2, \dots$ , where  $L(v_i) = 1$ . Then there are three possibilities:  $S_i = 0$ ,  $S_i = -n$  or  $S_i = n$ . Since labels  $\{1, 2, \dots, n\}$  are in cyclic order the labeling  $L$  is multivalued and it is defined up to  $n$ , i. e.  $L(v_j) = \ell_j + r_j n$  with  $r_j \in \mathbb{Z}$ . It is clear that  $r_i = q_i = 0, -1, 1$ .

So  $S_i = n$  if the sequence  $L_i := (L(v_1), L(v_2), \dots, L(v_i))$  makes a full cycle in the positive direction around labels  $(1, 2, \dots, n, 1)$ ,  $S_i = -n$  if it is a full cycle in the negative direction, and  $S_i = 0$  if it is not a full cycle.

We can consider the next index  $j$  with  $L(v_j) = 1$  and so on. Therefore,  $q_k = P_k - N_k$ , where  $P_k$  (respectively  $N_k$ ) is the number of full cycles in positive (respectively, in negative) direction in  $L_k$ .

For  $k = m$  we have  $S_m = q_m n$ . From the equality  $q_k = P_k - N_k$  follows that  $q_m = \deg([i, i+1], L)$  for all  $i$ .  $\square$

Since  $\deg([i, i+1], L)$  does not depend on  $i$ , denote  $\deg(L) := \deg([1, 2], L)$ .

**Remark 2.1.** Actually,  $\deg(L)$  is the degree of a piece-wise linear mapping  $f_L : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ , where  $f_L(v_i) = q_i$ ,  $v_i \in V \subset \mathbb{S}^1$  and  $\{q_1, \dots, q_n\}$  is a point set on  $\mathbb{S}^1$ . The degree of a continuous mapping between two compact oriented manifolds of the same dimension is a number that represents the number of times that the domain manifold wraps around the range manifold under the mapping (see for the case  $\mathbb{S}^1$  [12, Chapter 11]).

Consider a polygon  $M$  in the plane. Actually, we do not assume that  $M$  is a simple polygon, perhaps  $M$  is a *polygon with holes*, i. e.  $M$  enclosing several other polygons  $H_1, \dots, H_k$ . None of the boundaries of  $M$ ,  $H_1, \dots, H_k$  may intersect, and each the hole is empty.  $P$  is said to bound a multiply-connected region with  $k$  holes: the region of the plane interior to or on the boundary of  $M$ , but exterior to or on the boundary of  $H_1, \dots, H_k$ . (A polygon without holes is said to be simply-connected.)

Denote the outer boundary of  $M$  by  $H_0$ . Then the boundary of  $M$ , we denote it by  $\partial M$ , consists of  $H_0, \dots, H_k$ . It is well-known - a polygon with holes may be triangulated.

**Definition 2.2.** Let  $T$  be a triangulation of a polygon  $M$ . Consider a labeling  $L : V(T) \rightarrow \{1, 2, \dots, n\}$ , where  $V(T)$  denote the set of vertices of  $T$ . If  $\partial T$  is the disjoint union of polygons  $H_0, \dots, H_k$ , then  $L_i := L|_{H_i} : V(H_i) \rightarrow \{1, 2, \dots, n\}$  is a labelling on the circle  $H_i$ .

We write that  $L \in \text{NL}(T, n)$  if for all  $i = 0, \dots, k$  hold  $L_i \in \text{NL}(m_i, n)$ , where  $m_i = |V(H_i)|$  is the number of vertices of  $H_i$ .

Let vertices  $v_{01}, v_{02}, \dots, v_{0m_0}$  of the polygonal contour  $H_0$  are in counterclockwise order and vertices  $v_{i1}, v_{i2}, \dots, v_{im_i}$ ,  $i > 0$ , of  $H_i$  are in clockwise order. So  $M$  is a positively oriented polygon such that when traveling on the boundary vertices always the interior of  $M$  is to the left (and consequently,  $M$  exterior to the right).

Let  $L \in \text{NL}(T, n)$ . Now we define  $\deg(L, \partial T)$ .

$$\deg(L, \partial T) := \deg(L_0) + \deg(L_1) + \dots + \deg(L_k).$$

**Definition 2.3.** Let  $P$  be a set of  $n$  points  $v_1, \dots, v_n$  in the plane  $\mathbb{R}^2$ . Denote by  $S(P)$  the collection of all triangles spanned by vertices  $\{v_i, v_j, v_k\}$  with  $1 \leq i < j < k \leq n$ . Consider a point  $x \in \mathbb{R}^2$  and the set  $S_x(P)$  of all triangles from  $S(P)$  that cover  $x$ . If no such triangles exist, we write  $S_x(P) = \emptyset$ . Denote this set of triangles by  $\text{cov}_P(x)$  or just by  $\text{cov}(x)$ .

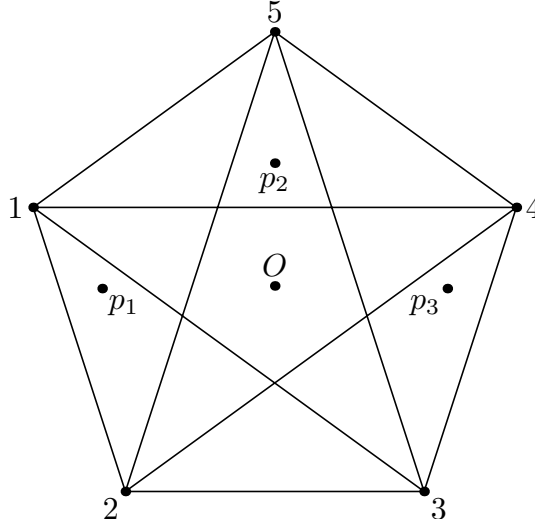


Figure 6: Pebbles and  $\text{cov}(x)$  for a pentagon

**Example.** Let  $P$  be a pentagon, see Fig. 6. Then

$$\text{cov}(p_1) = (123) \cup (124) \cup (125); \text{cov}(p_2) = (135) \cup (145) \cup (235) \cup (245);$$

$$\text{cov}(p_3) = (134) \cup (234) \cup (345); \text{cov}(O) = (124) \cup (134) \cup (135) \cup (235) \cup (245);$$

**Theorem 2.1.** Let  $P := \{y_1, \dots, y_n\}$  be a convex polygon. Let  $T$  be a triangulation of a planar polygon  $M$ . Then for any  $y \in P$  and a labelling  $L \in \text{NL}(T, n)$  the triangulation  $T$  must contain at least  $|\deg(L, \partial T)|$  triangles that are labelled as triangles in  $\text{cov}_P(y)$ .

*Proof.* Let  $v \in V(T)$ , i. e.  $v$  is a vertex of  $T$ . If  $L(v) = i$ , then we set  $f_L(v) := y_i$ . Therefore,  $f_L$  is defined for all vertices of  $T$ . Now we show that it defines a simplicial (piecewise linear) mapping  $f_L : M \rightarrow \mathbb{R}^2$ .

For every triangle  $t \in T$  with vertices  $u, v, w$  we have three correspondent points  $y_i, y_j, y_k$  in the plane. (Here  $i, j, k$  are not necessarily different.) It is well known that for given three non-collinear points in the plane there exists the unique linear (affine) transformation that carries these points to some other given three points in the plane. It defines  $f_L : t \rightarrow \mathbb{R}^2$  for all  $t$  and so  $f_L : M \rightarrow \mathbb{R}^2$  is well defined.

Let  $y$  be an internal point in  $P$ . Suppose that  $\text{cov}_P(y)$  does not contain triangles with  $y$  on its boundary, or equivalently, there are no edges  $y_i y_j$  that contain  $y$ . Then the set of preimages  $f_L^{-1}(y)$  is empty or consists of  $\{x_k\} \subset M$  such that every  $x_k$  lies inside of some triangle  $t_k \in T$  that is labelled as one of triangles in  $\text{cov}_P(y)$ . For all  $x_k$  can be assigned its signs. If  $f_L : t_k \rightarrow \mathbb{R}^2$  is orientation preserving, assign  $\text{sign}(x_k) := +1$ , and if it is orientation reversing, assign  $\text{sign}(x_k) := -1$ . If  $f_L^{-1}(y)$  is not empty set then  $\deg(L, y)$  is the sum of all  $\text{sign}(x_k)$ , otherwise it is 0.



Now we show that  $\deg(L, y)$  does not depend on  $y$  and is equal to  $\deg(L, \partial T)$ . Indeed, let us cut  $P$  by diagonals  $x_i x_j$  into connected components. Suppose  $p$  and  $q$  are points that belong to the same component. It is clear that  $f_L^{-1}([pq])$  consists of segments  $[uw]$  such that both  $u$  and  $w$  lie in some triangle of  $T$ . It immediately implies the equality  $\deg(L, p) = \deg(L, q)$ . It is easy to see that in the case when  $[pq] \subset P$  intersects only one diagonal  $x_i x_j$  the set of preimages  $f_L^{-1}([pq])$  can consist of three types of segments  $[u_1 u_2]$ ,  $[w_1 w_2]$  and  $[uw]$ , where  $f_L(u_1) = f_L(u_2) = f_L(u) = p$  and  $f_L(w_1) = f_L(w_2) = f_L(w) = q$ . Since  $\text{sign}(u_1) = -\text{sign}(u_2)$ ,  $\text{sign}(w_1) = -\text{sign}(w_2)$  and  $\text{sign}(u) = \text{sign}(w)$ , we have  $\deg(L, p) = \deg(L, q)$ . Thus, this equality holds for all  $p$  and  $q$  from  $P$  and if we take a point  $p$  that is closed to the edge  $y_1 y_2$ , then we get  $\deg(L, p) = \deg(L, \partial T)$ .  $\square$

**Remark 2.2.** We see that our proof of the theorem immediately follows from the fact that the degree of a continuous mapping does not depend on  $y$ . (Moreover, it is well known that the degree is a homotopy invariant.) So if for a space the degree of a mapping is well defined, then some version of this theorem holds, see Section 4.

### 3 Corollaries

Let  $P$  be a triangle in the plane. Take any internal point  $x \in P$ . It is clear that  $\text{cov}(x) = (123)$ . Therefore Theorem 2.1 yields

**Corollary 3.1.** *Let  $T$  be a triangulation of a planar polygon  $M$ . Then any labelling  $L : V(T) \rightarrow \{1, 2, 3\}$  must contain at least  $|\deg(L, \partial T)|$  fully colored triangles.*

**Corollary 3.2** (Sperner's lemma). *Any 3-Sperner labelling of a triangulation of a triangle contains a fully labelled triangle.*

*Proof.* It is easy to see that for any Sperner's labelling  $L$  we have  $|\deg(L, \partial T)| = 1$ . Thus, Corollary 3.1 implies Sperner's lemma.  $\square$

The following corollary extends the De Loera - Petersen - Su theorem in two dimensions.

**Corollary 3.3.** *Let  $T$  be a triangulation of a planar polygon  $M$ . Let  $L$  be a labelling such that  $L \in \text{NL}(T, n)$ . Then the number of fully labelled triangles in  $T$  is at least  $(n-2)|\deg(L, \partial T)|$ .*

*Proof.* Consider a set of points  $S$  in the interior of a convex  $n$ -gon  $P$  so that the interior of every triangle determined by three vertices of the polygon contains a unique point of  $S$ . In other words, for any two distinct points  $x$  and  $y$  in  $S$  the intersection of the sets  $\text{cov}_P(x)$  and  $\text{cov}_P(y)$  is empty. Such sets have been called *pebble sets* by De Loera, Peterson, and Su [3]. They considered this problem for all dimensions and particularly in dimension two they proved the existence of  $S$  with  $n-2$  points. Since any triangulation of  $P$  consists of  $n-2$  triangles this result cannot be improved.

(In Fig. 6 is shown a pebble set  $\{p_1, p_2, p_3\}$  in a pentagon. For pebble sets in convex polygons see also [4].)

Now take a convex  $n$ -gon  $P$  and a pebble set  $S = \{p_1, \dots, p_{n-2}\}$  in  $P$  of maximum size. For any  $p_i \in S$  from Theorem 2.1 follows that in  $T$  there are at least  $|\deg(L, \partial T)|$  fully labelled triangles. Since for distinct  $p_i$  these triangles have distinct labelling altogether we have at least  $(n-2)|\deg(L, \partial T)|$  fully labelled triangles.  $\square$

Now we show that from Theorem 2.1 follows Tucker's and Ky Fan's lemmas.

Let  $V := \{v_1, \dots, v_{2m}\}$ , where  $v_{i+m} = -v_i$ ,  $1 \leq i \leq m$ , be points in a circle. We call  $L : V \rightarrow \{\pm 1, \dots, \pm n\}$ ,  $n \geq 2$ , an *antipodal neighboring labelling* and write  $L \in \text{ANL}(2m, 2n)$  if  $L(v_{i+m}) = -L(v_i)$  and  $L$  is a neighboring labelling, i. e.  $L \in \text{NL}(2m, 2n)$ .

Note that  $L \in \text{ANL}(2m, 4)$  if and only if for any two neighbors  $u, v \in V$  we have  $L(u) \neq -L(v)$ . It can be extended for general case. If  $L \in \text{ANL}(2m, 2n)$ , then there is an antipodal piece-wise linear mapping  $f_L : \mathbb{S}^1 \rightarrow B$ , where  $B$  is the boundary of a centrally symmetric convex polygon in the plane with  $2n$  vertices  $y_1, \dots, y_n, -y_1, \dots, -y_n$ .

**Lemma 3.1.** *If  $L \in \text{ANL}(2m, 2n)$ , then  $\deg(L)$  is odd.*

*Proof.* Without loss of generality we may assume that  $L(v_1) = 1$ . So we have  $L(v_{m+1}) = -1$ . Let  $k_1$  is the minimum index such that  $L(v_{k_1}) = -1$ . Then  $1 < k_1 \leq m+1$ . If on the interval  $[k_1, m+1]$  there is  $i : L(v_i) = 1$  then denote by  $k_2$  the minimum such index. Let  $k_3$  is the minimum index such that  $L(v_{k_3}) = -1$  and  $k_2 < k_3 \leq m+1$ . And so on. Finally, we obtain the sequence  $k_0 = 1, k_1, \dots, k_\ell$ , where  $1 < k_1 < \dots < k_\ell \leq m+1$ . It is clear that  $\ell \geq 1$  and it is odd.

Consider two sequences:  $S_i := L(v_{k_i}), L(v_{k_i+1}), \dots, L(v_{k_{i+1}})$ , where  $i = 0, \dots, \ell-1$ , and  $S_{-i} := -S_i = L(v_{k_i+m}), L(v_{k_i+m+1}), \dots, L(v_{k_{i+1}+m})$ . Let  $R_i := S_i, S_{-i}$  is concatenation the two sequences. It is easy to see that  $\deg(R_i) = \pm 1$ . Since

$$\deg(L) = \sum_{i=0}^{\ell-1} \deg(R_i)$$

and  $\ell$  is odd, we have that  $\deg(L)$  is also odd.  $\square$

Corollary 3.3 and Lemma 3.1 yield the following extension of Tucker's lemma.

**Corollary 3.4.** *Let  $T$  be a triangulation of  $\mathbb{B}^2$  that antipodally symmetric on the boundary. Let  $L : V(T) \rightarrow \{+1, -1, +2, -2\}$  be a labelling that is antipodal on the boundary. Suppose there are no complementary edges on the boundary. Then there are at least  $|\deg(L, \partial T)|$  internal complementary edges. In particular, there is at least one.*

Now consider an extension of Ky Fan's lemma.

**Corollary 3.5.** *Let  $T$  be a triangulation of  $\mathbb{B}^2$  that antipodally symmetric on the boundary. Let*

$$L : V(T) \rightarrow \{+1, -1, +2, -2, \dots, +n, -n\}$$

be a labelling that is antipodal and NL on the boundary. Suppose this labelling does not have complementary edges. Then there are at least  $|\deg(L, \partial T)|$  triangles in  $T$  whose labels are of the form  $\{k_0, -k_1, k_2\}$ , where  $1 \leq |k_0| < |k_1| < |k_2| \leq n$  and all  $k_i$  have the same sign. In particular, there is at least one.

*Proof.* Here we apply Theorem 2.1. Let  $P$  be a centrally symmetric convex polygon in the plane with  $2n$  vertices  $y_1, \dots, y_n, y_{-1}, \dots, y_{-n}$ , where  $y_{-i} = -y_i$ . Let  $y = O$  is the center of  $P$ . Then  $\text{cov}_P(y)$  consists of edges  $y_i y_{-i}$ ,  $i = 1, \dots, n$ , and triangles in the form that is required in the theorem. Since there are no complementary edges we have only triangles.  $\square$

## 4 Generalizations of Sperner's lemma

Here we consider the main theorem and its corollaries for a very general class of spaces  $M$ . One of very natural extension of Section 2 is the case when  $M$  and  $P$  are polytopes in  $\mathbb{R}^d$ . In our papers [8, 9, 10] we studied a more general case when  $M$  is a piece-wise linear manifold. In this case, if  $M$  is a compact oriented manifold with boundary, then from the one side it extends Theorem 2.1 for all dimensions and to a huge class of spaces (even in two dimensions), on the other side, almost all proofs can be easily transfer for this case.

All results in this section hold for manifolds that admit triangulations. The class of such manifolds is called piece-wise linear (PL) manifolds. Note that a smooth manifold can be triangulated, therefore it is also a PL manifold. However, there are topological manifolds that do not admit a triangulation.

A *topological manifold* is a topological space that resembles Euclidean space near each point. More precisely, each point of a  $d$ -dimensional manifold has a neighbourhood that is homeomorphic to the Euclidean space of dimension  $d$ . A compact manifold without boundary is called *closed*. If a manifold contains its own boundary, it is called a *manifold with boundary*.

*Smooth manifolds* (also called *differentiable manifolds*) are manifolds for which overlapping charts “relate smoothly” to each other, meaning that the inverse of one followed by the other is an infinitely differentiable map from Euclidean space to itself.

$M$  is called a *piece-wise linear (PL) manifold* if it is a topological manifold together with a piecewise linear structure on it. Every PL manifold  $M$  admits a *triangulation*: that is, we can find a collection of simplices  $T$  of dimensions  $0, 1, \dots, d$ , such that (1) any face of a simplex belonging to  $T$  also belongs to  $T$ , (2) any nonempty intersection of any two simplices of  $T$  is a face of each, and (3) the union of the simplices of  $T$  is  $M$ . (See details in [2].) Actually, a PL-manifold  $M$  can be triangulated by many ways.

An *oriented simplex* is a simplex  $s$  together with a choice of one of its two possible orientations. In order to specify an orientation for  $s$  it suffices to list its vertices in some order since this ordering is a representative of precisely one of the two equivalence classes.

An *oriented PL-manifold*  $M$  of dimension  $d$  is a triangulation  $T$  of  $M$  equipped with a partial ordering on its vertices that restricts to an ordering on each  $d$ -simplex and such that any two  $d$ -simplices with a common  $(d - 1)$ -face must have the same orientation. Not all manifolds can be oriented. For instance, the Möbius strip is a non-orientable space.

Let  $T$  be a triangulation of a  $M$ . The vertex set of  $T$ , denoted by  $V(T)$ , is the union of the vertex sets of all simplices of  $T$ .

Given two triangulations  $T_1$  and  $T_2$  of two PL manifolds  $M_1$  and  $M_2$ . A *simplicial* map is a function  $f : V(T_1) \rightarrow V(T_2)$  that maps the vertices of  $T_1$  to the vertices of  $T_2$  and that has the property that for any simplex (face)  $s$  of  $T_1$ , the image set  $f(s)$  is a face of  $T_2$ .

We already discussed  $\deg(f)$  in Remarks 2.1 and 2.2. It is well known that the *degree of a continuous map* or *Brouwer's degree* is a topological invariant (see, for instance, [7] and [6, pp. 44–46]).

Let us define  $\deg(f)$  more rigorously. Let  $T_1$  be a triangulation of a closed  $d$ -dimensional oriented PL manifold  $M_1$ . Suppose that  $T_2$  is a triangulation of a  $d$ -dimensional oriented PL manifold  $M_2$ . Let  $f : V(T_1) \rightarrow V(T_2)$  be a simplicial map. Consider any  $d$ -simplex  $s$  of  $T_2$  and denote by  $\Pi(s)$  the set of preimages of  $s$  in  $T_1$ . Then any  $t \in \Pi(s)$  is a  $d$ -simplex in  $T_1$ . We have  $f(t) = s$  and  $H := f|_t : t \rightarrow s$  is a linear map. Then  $\det(H) \neq 0$  and the sign of this map is well defined. The sum of signs of all  $t \in \Pi(s)$  is  $\deg(f)$ . It can be proved that this number does not depend on  $s$ .

Let  $P$  be a convex polytope in  $\mathbb{R}^d$  with vertices  $\{p_1, \dots, p_n\}$ . Let  $T$  be a triangulation of a PL-manifold  $M$  of dimension  $d$ . Let  $L$  be an  $n$ -labelling of  $T$ ,  $L : V(T) \rightarrow \{1, 2, \dots, n\}$ . If for  $v \in V(T)$  we have  $L(v) = i$ , then set  $f_{L,P}(v) := p_i$ . Therefore,  $f_{L,P}$  is defined for all vertices of  $T$ , and it uniquely defines a simplicial (piecewise linear) map  $f_{L,P} : M \rightarrow \mathbb{R}^d$ .

Let  $\partial M$  and  $\partial P$  denote the boundary of  $M$  and the boundary of  $P$  in  $\mathbb{R}^d$  respectively. Suppose that  $f_{L,P}(\partial M) \subseteq \partial P$ . Then we have a mapping  $h := f_{L,P}|_{\partial M} : \partial M \rightarrow \partial P$  and  $\deg(h)$  is well defined. We denote it by  $\deg(L, \partial T)$ .

The following theorem can be proved by the same argument as Theorem 2.1.

**Theorem 4.1.** *Let  $P$  be a convex polytope in  $\mathbb{R}^d$  with  $n$  vertices. Let  $T$  be a triangulation of a compact oriented PL-manifold  $M$  of dimension  $d$ . Let  $L : V(T) \rightarrow \{1, 2, \dots, n\}$  be a labelling such that  $f_{L,P}(\partial M) \subseteq \partial P$ . Then for any  $y \in P$  the triangulation  $T$  must contain at least  $|\deg(L, \partial T)|$   $d$ -simplices that are labelled as simplices in  $\text{cov}_P(y)$ .*

Let  $P$  be a  $d$ -simplex in  $\mathbb{R}^d$ . Take any internal point  $y \in P$ . It is clear that  $\text{cov}(d) = (12 \dots d + 1)$ . Therefore Theorem 4.1 yields

**Corollary 4.1.** *Let  $T$  be a triangulation of a compact oriented PL-manifold  $M$  of dimension  $d$  with boundary. Then for any labelling  $L : V(T) \rightarrow \{1, 2, \dots, d + 1\}$  the triangulation  $T$  must contain at least  $|\deg(L, \partial T)|$  fully colored  $d$ -simplices.*

Since for any Sperner's labelling  $L$  we have  $|\deg(L, \partial T)| = 1$ , Corollary 4.1 implies Sperner's lemma.

The following corollary extends the De Loera - Petersen - Su theorem [3].

**Corollary 4.2.** *Let  $P$  be a convex polytope in  $\mathbb{R}^d$  with  $n$  vertices. Let  $T$  be a triangulation of a compact oriented PL-manifold  $M$  of dimension  $d$  with boundary. Let  $L : V(T) \rightarrow \{1, 2, \dots, n\}$  be a labelling such that  $f_{L,P}(\partial M) \subseteq \partial P$ . Then  $T$  contains at least  $(n - d)|\deg(L, \partial T)|$  fully labelled  $d$ -simplices.*

*Proof.* In [3] proved that in  $P$  there is a pebble set of cardinality at least  $n - d$ . Thus, Theorem 4.1 yields the corollary.  $\square$

Now consider extensions of Tucker's and Ky Fan's lemmas.

**Corollary 4.3.** *Let  $T$  be a triangulation of  $\mathbb{B}^d$  that antipodally symmetric on the boundary. Let  $L : V(T) \rightarrow \{+1, -1, \dots, +d, -d\}$  be a labelling that is antipodal on the boundary. Suppose there are no complementary edges on the boundary. Then there are at least  $|\deg(L, \partial T)|$  internal complementary edges. In particular, there is at least one.*

*Proof.* Let  $P$  be a centrally symmetric crosspolytope in  $\mathbb{R}^d$ . If there are no complementary edges on the boundary  $\partial T$ , then  $f_{L,P}(\partial M) \subseteq \partial P$ . Therefore,  $\deg(L, \partial T)$  is well defined and it is odd. Let  $0 \in P$  be the center of  $P$ . The corollary follows from Theorem 4.1 if we set  $y = 0$ .  $\square$

Let  $P$  be a convex polytope in  $\mathbb{R}^d$  with  $2n$  centrally symmetric vertices  $\{p_1, -p_1, \dots, p_n, -p_n\}$ . We say that  $P$  is ACS (Alternating Centrally Symmetric)  $(n, d)$ -polytope if the set of all simplices in  $\text{cov}_P(0)$ , that contain the origin  $0$  of  $\mathbb{R}^d$  inside, consists of edges  $(p_i, -p_i)$  and  $d$ -simplices with vertices  $\{p_{k_0}, -p_{k_1}, \dots, (-1)^d p_{k_d}\}$  and  $\{-p_{k_0}, p_{k_1}, \dots, (-1)^{d+1} p_{k_d}\}$ , where  $1 \leq k_0 < k_1 < \dots < k_d \leq n$ . In [9, Theorem 5.2] we proved that for any integer  $d \geq 2$  and  $n \geq d$  there exists ACS  $(n, d)$ -polytope.

**Corollary 4.4.** *Let  $P$  be an ACS  $(n, d)$ -polytope. Let  $T$  be a triangulation of  $\mathbb{B}^d$  that antipodally symmetric on the boundary. Let*

$$L : V(T) \rightarrow \{+1, -1, +2, -2, \dots, +n, -n\}$$

*be a labelling that is antipodal on the boundary and  $f_{L,P}(\partial T) \subseteq \partial P$ . Suppose this labelling does not have complementary edges. Then there are at least  $|\deg(L, \partial T)|$  simplices in  $T$  whose labels are of the form  $\{k_0, -k_1, \dots, (-1)^d k_d\}$ , where  $1 \leq |k_0| < |k_1| < \dots < |k_d| \leq n$  and all  $k_i$  have the same sign. In particular, there is at least one.*

*Proof.* Here we also apply Theorem 4.1. Let  $y = O$  is the center of  $P$ . Then  $\text{cov}_P(y)$  consists of edges  $y_i y_{-i}$ ,  $i = 1, \dots, n$ , and simplices in the form that is required in the theorem. Since there are no complementary edges we have only  $d$ -simplices.  $\square$

Theorem 4.1 and its corollaries can be also extended for the non-orientable case. This extension is based on the concept of the degree of a continuous mapping modulo 2. Let  $f : M_1 \rightarrow M_2$  be a continuous map between two closed manifolds  $M_1$  and  $M_2$  of the same dimension. The degree is a number that represents the number of times that the domain manifold wraps around the range manifold under the mapping. Then  $\deg_2(f)$  (the degree modulo 2) is 1 if this number is odd and 0 otherwise. It is well known that  $\deg_2(f)$  of a continuous map  $f$  is a topological invariant modulo 2.

**Theorem 4.2.** *Let  $P$  be a convex polytope in  $\mathbb{R}^d$  with  $n$  vertices. Let  $T$  be a triangulation of a PL-manifold  $M$  of dimension  $d$  with boundary. Let  $L : V(T) \rightarrow \{1, 2, \dots, n\}$  be a labelling such that  $f_{L,P}(\partial M) \subseteq \partial P$ . Suppose  $\deg_2(L, \partial T) \neq 0$ . Then for any  $y \in P$  the triangulation  $T$  must contain a simplex that is labelled as some simplex in  $\text{cov}_P(y)$ .*

**Corollary 4.5.** *Let  $T$  be a triangulation of a compact PL-manifold  $M$  of dimension  $d$  with boundary. Then any labelling  $L : V(T) \rightarrow \{1, 2, \dots, d+1\}$  with  $\deg_2(L, \partial T) \neq 0$  the triangulation  $T$  must contain at least one fully colored simplex.*

**Acknowledgment.** I wish to thank Arseniy Akopyan and Roman Karasev for helpful discussions and comments.

## References

- [1] K. T. Atanassov, On Sperner's lemma, *Studia Sci. Math. Hungar.*, **32** (1996), 71-74.
- [2] J. L. Bryant, Piecewise linear topology, *Handbook of geometric topology*, 219-259, North-Holland, Amsterdam, 2002.
- [3] J. A. De Loera, E. Peterson, and F. E. Su, A Polytopal Generalization of Sperner's Lemma, *J. of Combin. Theory Ser. A*, **100** (2002), 1-26.
- [4] K. Iga and R. Maddox, Pebble Sets in Convex Polygons. *Discrete & Computational Geometry*, **38** (2007), 680-700.
- [5] K. Fan, A generalization of Tucker's combinatorial lemma with topological applications. *Ann. of Math.*, **56** (1952), 431-437.
- [6] J. Matoušek, Using the Borsuk-Ulam theorem, Springer-Verlag, Berlin, 2003.
- [7] J. W. Milnor, Topology from the differentiable viewpoint, The University Press of Virginia, Charlottesville, Virginia, 1969.
- [8] O. R. Musin, Borsuk-Ulam type theorems for manifolds, *Proc. Amer. Math. Soc.* **140** (2012), 2551-2560.
- [9] O. R. Musin, Extensions of Sperner and Tucker's lemma for manifolds, arXiv:1212.1899.
- [10] O. R. Musin, KKM and Kakutani type Borsuk-Ulam theorems, arXiv:1405.1579
- [11] T. Prescott and F. E. Su. A constructive proof of Ky Fan's generalization of Tucker's lemma, *J. Combin. Theory Ser. A*, **111** (2005), 257-265.
- [12] Yu. A. Shashkin, Fixed Points, American Mathematical Society, Providence, RI, 1991.

- [13] E. Sperner, Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes, Abh. Math. Sem. Univ. Hamburg **6** (1928), 265-272.
- [14] A. W. Tucker, Some topological properties of the disk and sphere. In: Proc. of the First Canadian Math. Congress, Montreal, 285-309, 1945.

O. R. Musin

Department of Mathematics, University of Texas at Brownsville, One West University Boulevard, Brownsville, TX, 78520

and

IITP RAS, Bolshoy Karetny per. 19, Moscow, 127994, Russia

*E-mail address:* oleg.musin@utb.edu